

AMARI-CHENTSOV CONNECTIONS AND THEIR GEODESICS ON HOMOGENEOUS SPACES OF DIFFEOMORPHISM GROUPS

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ABSTRACT. We study the family of α -connections of Amari-Chentsov on the homogeneous space $\mathcal{D}(M)/\mathcal{D}_\mu(M)$ of diffeomorphisms modulo volume-preserving diffeomorphisms of a compact manifold M . We show that in some cases their geodesic equations yield completely integrable Hamiltonian systems.

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1. INTRODUCTION

In a recent paper [5] several key notions of geometric statistics were developed on diffeomorphism groups and their quotient spaces equipped with right-invariant metrics. For example, the metric defined by (the homogeneous part of) the Sobolev H^1 inner product of vector fields on the underlying compact manifold was shown to induce on the quotient of the diffeomorphism group by its subgroup of volume-preserving diffeomorphisms an infinite-dimensional analogue of the Fisher-Rao metric while geodesics of its Levi-Civita connection - when the underlying manifold is the circle - were shown to be related to solutions of a well-known one-dimensional completely integrable equation [6, 7]. Furthermore, the authors also described analogues of the so-called α -connections introduced in geometric statistics by Chentsov [3] and Amari [1] and pointed out integrability of another geodesic equation corresponding to $\alpha = -1$.

Our goals in this paper are to provide a proof of the construction in [5] of Amari-Chentsov connections for circle diffeomorphisms (Theorem 2.1), to generalize this construction to diffeomorphism groups of higher-dimensional manifolds (Theorem 3.1) and finally, as a by-product, to show integrability of the geodesic equations corresponding to $\alpha = 1$ (Theorem 2.6, Corollary 3.3).

1.1. Diffeomorphism groups and the Fisher-Rao metric. Let M be a compact Riemannian manifold without boundary. Let $\mathcal{D}(M)$ denote the group of smooth diffeomorphisms of M and let $\mathcal{D}_\mu(M)$ be its subgroup of diffeomorphisms preserving the volume form μ on M . It is well-known that the completion of $\mathcal{D}(M)$ (resp. $\mathcal{D}_\mu(M)$) in the H^s Sobolev norm with $s > n/2 + 1$ can be equipped with the structure of a smooth Hilbert manifold whose tangent space at the identity diffeomorphism e consists of all H^s vector fields (resp. all divergence-free H^s vector fields) on M . However, for what follows it will be sufficient to work with the smooth category. It will also be convenient to normalize the Riemannian volume $\mu(M) = 1$.

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For any $\eta \in \mathcal{D}(M)$ and any $V, W \in T_\eta \mathcal{D}(M)$ we set

$$(1.1) \quad \langle V, W \rangle_{\dot{H}^1} = \frac{1}{4} \int_M \operatorname{div} v \operatorname{div} w \, d\mu$$

where $V = v \circ \eta$ and $W = w \circ \eta$ with $v, w \in T_e \mathcal{D}(M)$ to obtain a right-invariant (degenerate) \dot{H}^1 metric on $\mathcal{D}(M)$.

The geometry of this metric turns out to be particularly remarkable. The homogeneous space $\mathcal{D}(M)/\mathcal{D}_\mu(M)$ can be naturally identified with the set of smooth probability densities, i.e., smooth functions $\rho > 0$ on M satisfying the condition $\int_M \rho \, d\mu = 1$. If N is an n -dimensional submanifold of such densities $\rho = \rho_{t_1 \dots t_n}$ parameterized by $(t_1, \dots, t_n) \in \mathbb{R}^n$ then recall that the Fisher-Rao metric on N is given by the formula

$$(1.2) \quad g_{ij} = \int_M \frac{\partial \log \rho}{\partial t_i} \frac{\partial \log \rho}{\partial t_j} \rho \, d\mu \quad 1 \leq i, j \leq n.$$

It turns out that the right-invariant \dot{H}^1 metric defined by (1.1) on $\mathcal{D}(M)$ descends to a (nondegenerate) metric on $\mathcal{D}(M)/\mathcal{D}_\mu(M)$ and the map $\eta \mapsto \sqrt{\operatorname{Jac}_\mu \eta}$ defines an isometry between $\mathcal{D}(M)/\mathcal{D}_\mu(M)$ and a subset of the unit sphere in $L^2(M, d\mu)$ with the canonical round metric. Furthermore, the restriction of the \dot{H}^1 metric to any finite-dimensional submanifold N of $\mathcal{D}(M)/\mathcal{D}_\mu(M)$ coincides with the Fisher-Rao metric (1.2) on N while its Riemannian distance is the spherical Hellinger distance between probability densities on M . Proofs of these statements can be found in Section 3 of [5].

Thus, in the framework of diffeomorphism groups, information geometry associated with the Fisher-Rao metric and its spherical Hellinger distance can be viewed as an \dot{H}^1 analogue of standard optimal transport associated with the metric¹ on $\mathcal{D}(M)/\mathcal{D}_\mu(M)$ induced by the (non-invariant) L^2 metric on $\mathcal{D}(M)$ and whose Riemannian distance is the celebrated Kantorovich (or Wasserstein) distance, cf. [9].

1.2. Divergence functions and dual connections. Recall that a divergence on an n -dimensional manifold N is a smooth function $D : N \times N \rightarrow \mathbb{R}$ satisfying $D(p||q) \geq 0$ with equality if and only if $p = q$ and such that the matrix g_{ij}^D defined in a chart at $p \in N$ by

$$(1.3) \quad g_{ij}^D(p) = -\frac{\partial}{\partial p_i} \frac{\partial}{\partial q_j} D(p||q)|_{p=q} \quad 1 \leq i, j \leq n$$

is strictly positive definite for every $p \in N$. Equation (1.3) defines a Riemannian metric on N with covariant derivative determined by

$$(1.4) \quad \Gamma_{ij,k}^D = -\frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} \frac{\partial}{\partial q_k} D(p||q)|_{p=q} \quad 1 \leq i, j, k \leq n.$$

In what follows we shall consider on $\mathcal{D}(M) \times \mathcal{D}(M)$ the functions

$$(1.5) \quad D^{(\alpha)}(\xi||\eta) = \frac{1}{1-\alpha^2} \left(1 - \int_M (\operatorname{Jac}_\mu \xi)^{\frac{1-\alpha}{2}} (\operatorname{Jac}_\mu \eta)^{\frac{1+\alpha}{2}} \, d\mu \right), \quad -1 < \alpha < 1$$

$$(1.6) \quad D^{(-1)}(\xi||\eta) = D^{(1)}(\eta||\xi) = \frac{1}{4} \int_M (\log \operatorname{Jac}_\mu \xi - \log \operatorname{Jac}_\mu \eta) \operatorname{Jac}_\mu \xi \, d\mu.$$

¹This metric is sometimes called Otto's metric.

These functions are well-defined on the homogeneous space $\mathcal{D}(M)/\mathcal{D}_\mu(M)$ and satisfy $D^{(\alpha)}(\xi||\eta) \geq 0$ with equality if and only if ξ and η project onto the same probability density on M .

Furthermore, recall that two affine connections ∇ and ∇^* on a Riemannian manifold N are called dual (or conjugate) relative to the Riemannian metric if for any vector fields U , V and W they satisfy $W\langle U, V \rangle = \langle \nabla_W U, V \rangle + \langle U, \nabla_W^* V \rangle$. Basic facts about dual connections and divergences can be found in [1] or [4].

2. THE ONE-DIMENSIONAL CASE: $\mathcal{D}(S^1)/\text{Rot}(S^1)$

We first consider the case when the underlying manifold is the circle $S^1 = \mathbb{R}/\mathbb{Z}$. In this case $\mathcal{D}_\mu(S^1)$ is the space of rigid rotations $\text{Rot}(S^1) \simeq S^1$. It will be convenient to identify the homogeneous space $\mathcal{D}(S^1)/\text{Rot}(S^1)$ with the subgroup of circle diffeomorphisms which fix a prescribed point, for example with $\{\eta \in \mathcal{D}(S^1) : \eta(0) = 0\}$. Its tangent space at the identity can be identified with the space of smooth periodic functions vanishing at 0. Furthermore, given any such function $u(x)$ we define the operator

$$(2.1) \quad A^{-1}u(x) = - \int_0^x \int_0^y u(z) dz dy + x \int_0^1 \int_0^y u(z) dz dy$$

i.e., the inverse of $A = -\partial_x^2$.

The following is a reformulation of Theorem 6.2 stated (without proof) in [5]. Our first objective is to provide a proof of this result.

Theorem 2.1 (cf. [5], Section 6).

- (i) *Each divergence $D^{(\alpha)}$ induces on $\mathcal{D}(S^1)/\text{Rot}(S^1)$ the \dot{H}^1 metric and an affine connection $\nabla^{(\alpha)}$ whose Christoffel symbols are given by*

$$(2.2) \quad \Gamma_\eta^{(\alpha)}(W, V) = -\frac{1+\alpha}{2} \left\{ A^{-1} \partial_x [(V \circ \eta^{-1})_x (W \circ \eta^{-1})_x] \right\} \circ \eta, \quad -1 \leq \alpha \leq 1.$$

- (ii) *For any α the connections $\nabla^{(\alpha)}$ and $\nabla^{(-\alpha)}$ are dual with respect to the \dot{H}^1 metric and $\nabla^{(0)}$ is the self-dual Levi-Civita connection.*
- (iii) *The geodesic equation of $\nabla^{(\alpha)}$ on $\mathcal{D}(S^1)/\text{Rot}(S^1)$ is the generalized Proudman-Johnson equation*

$$(2.3) \quad u_{txx} + (2 - \alpha)u_x u_{xx} + u u_{xxx} = 0.$$

In particular, $\alpha = 0$ yields the completely integrable Hunter-Saxton equation

$$(2.4) \quad u_{txx} + 2u_x u_{xx} + u u_{xxx} = 0$$

and $\alpha = -1$ yields the completely integrable μ -Burgers equation

$$(2.5) \quad u_{txx} + 3u_x u_{xx} + u u_{xxx} = 0.$$

Remark 2.2. The equation corresponding to $\alpha = 1$ also turns out to be integrable and its solutions can be given explicitly, see Theorem 2.6 below.

Proof. The metrics induced by $D^{(\alpha)}$ and their connections can be calculated essentially as in finite dimensions using formulas (1.3) and (1.4). We first assume that $\alpha \neq \pm 1$. Given any tangent vectors V and W at $\eta \in \mathcal{D}(S^1)$ let $\eta(s, t)$ be

a two-parameter family of diffeomorphisms in $\mathcal{D}(S^1)$ such that $\eta(0,0) = \eta$ with $\frac{\partial}{\partial s}\eta(0,0) = V$ and $\frac{\partial}{\partial t}\eta(0,0) = W$. From (1.3) and (1.5) we have

$$\begin{aligned}
 \langle V, W \rangle_\alpha &= -\frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}D^{(\alpha)}(\eta(s,0)\|\eta(0,t)) \\
 (2.6) \quad &= \frac{1}{1-\alpha^2}\frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}\int_{S^1}\eta_x(s,0)^{\frac{1-\alpha}{2}}\eta_x(0,t)^{\frac{1+\alpha}{2}}dx \\
 &= \frac{1}{4}\int_{S^1}V_xW_x\eta_x^{\frac{-1-\alpha}{2}}\eta_x^{\frac{-1+\alpha}{2}}dx = \frac{1}{4}\int_{S^1}\frac{V_xW_x}{\eta_x}dx \\
 &= \langle V, W \rangle_{\dot{H}^1}.
 \end{aligned}$$

Now suppose that W is a vector field on $\mathcal{D}(S^1)$ defined in a neighborhood of η . Let $\eta(s,t,r)$ be a three-parameter family of diffeomorphisms such that $\eta(0,0,0) = \eta$ with $\frac{\partial}{\partial s}\eta(0,0,0) = V$, $\frac{\partial}{\partial t}\eta(s,0,0) = W_{\eta(s,0,0)}$ for all sufficiently small s , and $\frac{\partial}{\partial r}\eta(0,0,0) = Z$. It is clear that such a map $\eta(s,t,r)$ exists. From the formulas (1.4), (1.5) and (2.6) we have

$$\begin{aligned}
 \langle \nabla_V^{(\alpha)}W, Z \rangle_\alpha &= \frac{1}{4}\int_{S^1}\frac{(\nabla_V^{(\alpha)}W)_xZ_x}{\eta_x}dx \\
 &= -\frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial r}\Big|_{r=0}D^{(\alpha)}(\eta(s,t,0)\|\eta(0,0,r)) \\
 &= \frac{1}{1-\alpha^2}\frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial r}\Big|_{r=0}\int_{S^1}\eta_x(s,t,0)^{\frac{1-\alpha}{2}}\eta_x(0,0,r)^{\frac{1+\alpha}{2}}dx \\
 (2.7) \quad &= \frac{1}{4}\frac{\partial}{\partial s}\Big|_{s=0}\int_{S^1}W_x(\eta(s,0,0))\eta_x(s,0,0)^{\frac{-1-\alpha}{2}}Z_x\eta_x^{\frac{-1+\alpha}{2}}dx \\
 &= \frac{1}{4}\int_{S^1}(DW \cdot V)_xZ_x\eta_x^{-1}dx - \frac{1+\alpha}{8}\int_{S^1}W_xV_x\eta_x^{\frac{-3-\alpha}{2}}Z_x\eta_x^{\frac{-1+\alpha}{2}}dx \\
 &= \frac{1}{4}\int_{S^1}\left\{((DW \cdot V) \circ \eta^{-1})_x - \frac{1+\alpha}{2}(V \circ \eta^{-1})_x(W \circ \eta^{-1})_x\right\}(Z \circ \eta^{-1})_xdx
 \end{aligned}$$

and integrating by parts and using the fact that Z is arbitrary we find that

$$(\nabla_V^{(\alpha)}W)_\eta = (DW \cdot V)(\eta) - \Gamma_\eta^\alpha(W, V)$$

where the Christoffel map is given by (2.2).

We will use the same notation for calculations in the remaining cases $\alpha = \pm 1$. From (1.1) and (1.6) we have

$$\begin{aligned}
 \langle V, W \rangle_{-1} &= \langle V, W \rangle_1 = -\frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}D^{(-1)}(\eta(s,0)\|\eta(0,t)) \\
 (2.8) \quad &= -\frac{1}{4}\frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}\int_{S^1}\eta_x(s,0)(\log \eta_x(s,0) - \log \eta_x(0,t))dx \\
 &= \frac{1}{4}\frac{\partial}{\partial s}\Big|_{s=0}\int_{S^1}\eta_x(s,0)\frac{W_x}{\eta_x}dx = \frac{1}{4}\int_{S^1}\frac{V_xW_x}{\eta_x}dx \\
 &= \langle V, W \rangle_{\dot{H}^1}.
 \end{aligned}$$

The corresponding affine connections can be obtained as in (2.7) using (1.4), (1.6) and (2.8). When $\alpha = -1$ we have

$$\begin{aligned} \int_{S^1} \frac{(\nabla_V^{(-1)} W)_x Z_x}{\eta_x} dx &= -\frac{\partial}{\partial s} \Big|_{s=0} \frac{\partial}{\partial t} \Big|_{t=0} \frac{\partial}{\partial r} \Big|_{r=0} \int_{S^1} \eta_x(s, t, 0) \log \frac{\eta_x(s, t, 0)}{\eta_x(0, 0, r)} dx \\ &= \frac{\partial}{\partial s} \Big|_{s=0} \frac{\partial}{\partial t} \Big|_{t=0} \int_{S^1} \eta_x(s, t, 0) \frac{Z_x}{\eta_x} dx \\ &= \frac{\partial}{\partial s} \Big|_{s=0} \int_{S^1} W_x(s, 0, 0) \frac{Z_x}{\eta_x} dx \\ &= \int_{S^1} (DW \cdot V)_x \frac{Z_x}{\eta_x} dx \end{aligned}$$

from which we deduce that

$$(2.9) \quad \Gamma_\eta^{(-1)}(W, V) = 0.$$

When $\alpha = 1$ an analogous calculation gives

$$(2.10) \quad \Gamma_\eta^{(1)}(W, V) = -A^{-1} \partial_x \{ (V \circ \eta^{-1})_x (W \circ \eta^{-1})_x \} \circ \eta.$$

This establishes the first part of the theorem.

For the second part we need to verify that for any vector fields X, Y and Z on $\mathcal{D}(S^1)/\text{Rot}(S^1)$ we have

$$(2.11) \quad X \langle Y, Z \rangle_{\dot{H}^1} = \langle \nabla_X^{(\alpha)} Y, Z \rangle_{\dot{H}^1} + \langle Y, \nabla_X^{(-\alpha)} Z \rangle_{\dot{H}^1}.$$

This can be done either by a direct calculation as above or else it can be deduced from general properties of divergences of the type (1.5) and (1.6) which are discussed in Chapter 3 of [1]. The fact that $\nabla^{(0)}$ is the Levi-Civita connection of the \dot{H}^1 metric follows at once from (2.11).

The equation for geodesics of $\nabla^{(\alpha)}$ on $\mathcal{D}(S^1)/\text{Rot}(S^1)$ has the form

$$\frac{d^2 \gamma}{dt^2} = \Gamma_\gamma^{(\alpha)} \left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right).$$

Setting $d\gamma/dt = u \circ \gamma$ defines a time-dependent vector field u on the circle S^1 (i.e., a periodic function vanishing at $x = 0$). Differentiating this relation with respect to t and eliminating the first and second derivatives of γ from the geodesic equation gives

$$(u_t + uu_x) \circ \gamma = \Gamma_\gamma^{(\alpha)}(u \circ \gamma, u \circ \gamma).$$

Using (2.2) and composing both sides with γ^{-1} we obtain a nonlinear pseudodifferential equation

$$u_t + uu_x = -\frac{1+\alpha}{2} A^{-1} \partial_x (u_x^2)$$

which we can rewrite as a nonlinear PDE

$$-u_{txx} - 3u_x u_{xx} - uu_{xxx} = -(1+\alpha)u_x u_{xx}$$

yielding (2.3). □

Remark 2.3. The Hunter-Saxton equation (2.4) can be alternatively derived by observing that it is the Euler-Arnold equation of $\nabla^{(0)}$ on $T_e \mathcal{D}(S^1)/\text{Rot}(S^1)$ and as such it is obtained from the geodesic equation of the right-invariant \dot{H}^1 metric (1.1) by a standard reduction procedure, see [6].

Remark 2.4. Another form of the Proudman-Johnson equation can be obtained by integrating (2.3) in the x variable

$$u_{tx} + uu_{xx} + \frac{1-\alpha}{2} u_x^2 = C(t)$$

where $C(t) = -\frac{1+\alpha}{2} \int_{S^1} u_x^2 dx$. Observe that $C(t)$ is a conserved integral of the equation (2.4) when $\alpha = 0$.

Remark 2.5 (α -curvature). Using the Christoffel symbols (2.2) it is possible to calculate the curvature of the α -connections. It turns out to be proportional to the curvature of the \dot{H}^1 metric, i.e. for any vector fields X, Y and Z on $\mathcal{D}(S^1)/\text{Rot}(S^1)$ we have

$$(2.12) \quad R^{(\alpha)}(X, Y)Z = (1 - \alpha^2) \left(X \langle Y, Z \rangle_{\dot{H}^1} + Y \langle X, Z \rangle_{\dot{H}^1} \right).$$

This formula can be computed as in finite dimensions; see [8] where a different choice of parameters is made.

It turns out that the geodesic equation corresponding to (2.3) with $\alpha = 1$ can be integrated as well, albeit indirectly, by constructing affine coordinates for $\nabla^{(1)}$. Observe that from (2.12) we already know that the connections $\nabla^{(-1)}$ and $\nabla^{(1)}$ are flat. In the former case this is also evident from (2.9).

Theorem 2.6. *The geodesic equation*

$$(2.13) \quad u_{txx} + u_x u_{xx} + uu_{xxx} = 0$$

of $\nabla^{(1)}$ is integrable. Its general solution is given by

$$(2.14) \quad u = \frac{d\eta}{dt} \circ \eta^{-1} \quad \text{where} \quad \eta(t, x) = \frac{\int_0^x e^{a(y)t+b(y)} dy}{\int_{S^1} e^{a(x)t+b(x)} dx}$$

and where a, b are smooth mean-zero functions on S^1 .

Proof. We will construct a chart on $\mathcal{D}(S^1)/\text{Rot}(S^1)$ in which the Christoffel symbols of $\nabla^{(1)}$ vanish identically. Consider the map

$$(2.15) \quad \eta \mapsto \phi(\eta) = \log \eta_x - \int_{S^1} \log \eta_x dx$$

from $\mathcal{D}(S^1)/\text{Rot}(S^1)$ to the space of smooth periodic mean-zero functions. To see how the Christoffel symbols transform under $\eta \mapsto \tilde{\eta} = \phi(\eta)$ we first compute the derivatives

$$D_\eta \phi(W) = \frac{W_x}{\eta_x} - \int_{S^1} \frac{W_x}{\eta_x} dx, \quad D_\eta^2 \phi(W, V) = -\frac{V_x W_x}{\eta_x^2} + \int_{S^1} \frac{V_x W_x}{\eta_x^2} dx$$

for any $V, W \in T_\eta \mathcal{D}(S^1)/\text{Rot}(S^1)$. Next, from (2.1) and (2.10) we obtain

$$\tilde{\Gamma}_{\phi(\eta)}^{(1)}(D_\eta \phi(W), D_\eta \phi(V)) = D_\eta^2 \phi(W, V) + D_\eta \phi(\Gamma_\eta^{(1)}(W, V))$$

$$\begin{aligned}
&= -\frac{V_x W_x}{\eta_x^2} + \int_{S^1} \frac{V_x W_x}{\eta_x^2} dx - \frac{(A^{-1}(v_x w_x)_x \circ \eta)_x}{\eta_x} + \int_{S^1} \frac{(A^{-1}(v_x w_x)_x \circ \eta)_x}{\eta_x} dx \\
&= -(v_x w_x) \circ \eta + \int_{S^1} (v_x w_x) \circ \eta dx - (A^{-1}(v_x w_x)_x)_x \circ \eta + \int_{S^1} (A^{-1}(v_x w_x)_x)_x \circ \eta dx \\
&= -(v_x w_x) \circ \eta + \int_{S^1} (v_x w_x) \circ \eta dx - \left(-v_x w_x + \int_{S^1} v_x w_x dx \right) \circ \eta \\
&\quad + \int_{S^1} \left(-v_x w_x + \int_{S^1} v_x w_x dx \right) \circ \eta dx = 0,
\end{aligned}$$

where $v = V \circ \eta^{-1}$ and $w = W \circ \eta^{-1}$.

We can now solve (2.13) as follows. Since $\tilde{\Gamma}^{(1)} \equiv 0$ all geodesics of $\nabla^{(1)}$ in the affine coordinates are the straight lines which can be written as

$$t \rightarrow \tilde{\eta}(t, x) = a(x)t + b(x) \quad x \in S^1$$

for some smooth mean-zero periodic functions a and b . Thus, given any such functions to construct a general solution u it is sufficient to (i) invert the map ϕ in (2.15) to obtain the flow $t \rightarrow \eta(t) = \phi^{-1}\tilde{\eta}(t)$ and (ii) right-translate the velocity vector of $\eta(t)$ to the tangent space at the identity in $\mathcal{D}(S^1)/\text{Rot}(S^1)$.² The required formulas are those in (2.14). \square

It is worth pointing out that the above proof manifests integrability of (2.13) in that it provides an explicit change of coordinates that linearizes the flow in the same spirit as the inverse scattering transform formalism.

3. THE n -DIMENSIONAL CASE: $\mathcal{D}(M)/\mathcal{D}_\mu(M)$

We now turn to the general case when M is an n -dimensional compact Riemannian manifold without boundary and work with the coset space $\mathcal{D}(M)/\mathcal{D}_\mu(M)$. Our next result is stated in analogy with Theorem 2.1.

Theorem 3.1.

- (i) *Each divergence $D^{(\alpha)}$ induces on the quotient $\mathcal{D}(M)/\mathcal{D}_\mu(M)$ the \dot{H}^1 metric (1.1) and an affine connection $\nabla^{(\alpha)}$ given for a right-invariant vector field $W_\eta = w \circ \eta$ and a tangent vector $V = v \circ \eta$ by*

$$(3.1) \quad (\nabla_V^{(\alpha)} W)_\eta = - \left\{ \Delta^{-1} d \left(d \operatorname{div} w \cdot v + \frac{1-\alpha}{2} \operatorname{div} w \operatorname{div} v \right) \right\}^\# \circ \eta$$

where $\alpha \in [-1, 1]$ and $\Delta = d\delta + \delta d$ denotes the Laplace-de Rham operator.

- (ii) *For any α the connections $\nabla^{(\alpha)}$ and $\nabla^{(-\alpha)}$ are dual with respect to the \dot{H}^1 metric and $\nabla^{(0)}$ is the Levi-Civita connection.*
(iii) *The geodesic equation of $\nabla^{(\alpha)}$ on $\mathcal{D}(M)/\mathcal{D}_\mu(M)$ is equivalent to the following nonlinear PDE*

$$(3.2) \quad d\varphi_t + d\iota_u d\varphi + (1-\alpha)\varphi d\varphi = 0, \quad \varphi = \operatorname{div} u.$$

²In the language of fluid dynamics the second step corresponds to going from Lagrangian to Eulerian coordinates.

Proof. Despite the fact that we are now working with cosets the computations involved in the proof of statements (i) and (ii) are similar to the one-dimensional case in Section 2. We will prove (i) for $\alpha \in (-1, 1)$; the proofs for $\alpha = \pm 1$ are analogous and will be omitted.

Let W be a right-invariant vector field on $\mathcal{D}(M)$ defined in a neighborhood of $\eta \in \mathcal{D}(M)$. Given $V, Z \in T_\eta \mathcal{D}(M)$ let $\eta(s, t, r)$ be a three-parameter family of diffeomorphisms such that $\eta(0, 0, 0) = \eta$ with $\frac{\partial}{\partial s} \eta(0, 0, 0) = V$, $\frac{\partial}{\partial t} \eta(s, 0, 0) = W_{\eta(s, 0, 0)}$ for all sufficiently small s and $\frac{\partial}{\partial r} \eta(0, 0, 0) = Z$. Using the identity

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} \text{Jac}_\mu \eta(s, t, 0) &= \text{div}(W_{\eta(s, 0, 0)} \circ \eta^{-1}(s, 0, 0)) \circ \eta(s, 0, 0) \text{Jac}_\mu \eta(s, 0, 0) \\ &= \text{div } w \circ \eta(s, 0, 0) \text{Jac}_\mu \eta(s, 0, 0), \end{aligned}$$

where $W_\eta = w \circ \eta$ and similar computations for the partial derivatives in s and r we have

$$\begin{aligned} \langle \nabla_V^{(\alpha)} W, Z \rangle_\alpha &= \frac{1}{4} \int_M \text{div}((\nabla_V^{(\alpha)} W) \circ \eta^{-1}) \text{div}(Z \circ \eta^{-1}) d\mu \\ &= -\frac{\partial}{\partial s} \Big|_{s=0} \frac{\partial}{\partial t} \Big|_{t=0} \frac{\partial}{\partial r} \Big|_{r=0} D^{(\alpha)}(\eta(s, t, 0) \parallel \eta(0, 0, r)) \\ &= \frac{1}{1 - \alpha^2} \frac{\partial}{\partial s} \Big|_{s=0} \frac{\partial}{\partial t} \Big|_{t=0} \frac{\partial}{\partial r} \Big|_{r=0} \int_M (\text{Jac}_\mu \eta(s, t, 0))^{\frac{1-\alpha}{2}} (\text{Jac}_\mu \eta(0, 0, r))^{\frac{1+\alpha}{2}} d\mu \\ &= \frac{1}{4} \frac{\partial}{\partial s} \Big|_{s=0} \int_M \text{div } w \circ \eta(s, 0, 0) (\text{Jac}_\mu \eta(s, 0, 0))^{\frac{1-\alpha}{2}} \text{div}(Z \circ \eta^{-1}) \circ \eta (\text{Jac}_\mu \eta)^{\frac{1+\alpha}{2}} d\mu \\ &= \frac{1}{4} \int_M \left\{ d \text{div } w \circ \eta \cdot V \right. \\ &\quad \left. + \frac{1-\alpha}{2} \text{div } w \circ \eta \text{div}(V \circ \eta^{-1}) \circ \eta \right\} \text{div}(Z \circ \eta^{-1}) \circ \eta \text{Jac}_\mu \eta d\mu \\ &= \frac{1}{4} \int_M \left\{ d \text{div } w \cdot v + \frac{1-\alpha}{2} \text{div } w \text{div } v \right\} \text{div}(Z \circ \eta^{-1}) d\mu, \end{aligned}$$

where $V = v \circ \eta$. Using integration by parts

$$\int_M f \text{div } X d\mu = - \int_M f \delta X^\flat d\mu = - \int_M df(X) d\mu$$

and the fact that Z is arbitrary, we find

$$d \text{div} \{(\nabla_V^{(\alpha)} W) \circ \eta^{-1}\} = df \quad \text{i.e.} \quad \text{div} \{(\nabla_V^{(\alpha)} W) \circ \eta^{-1}\} = f - \int_M f d\mu,$$

where $f = d \text{div } w \cdot v + \frac{1-\alpha}{2} \text{div } w \text{div } v$. The expression in (3.1) now follows from the next lemma.

Lemma 3.2. *Let Δ be the Laplace-de Rham operator. Then*

$$\text{div}(-(\Delta^{-1} df)^\sharp) = f - \int_M f d\mu, \quad f \in C^\infty(M).$$

Proof. If $g = \text{div}(-(\Delta^{-1} df)^\sharp) = \delta \Delta^{-1} df$ then $\Delta g = \delta df = \Delta f$ and since the kernel of Δ acting on functions consists of the constants, we deduce that $g = f + c$ for some constant c . Integrating over M yields $0 = \int_M f d\mu + c$, which determines c . \square

Calculation of $\nabla_V^{(\alpha)} W$ for an arbitrary (not necessarily right-invariant) vector field W can be reduced to the right-invariant case as follows. Let $\eta \in \mathcal{D}(M)$ and let W^R be the right-invariant vector field with $W_\eta^R = W_\eta$. Then

$$(3.3) \quad (\nabla_V^{(\alpha)} W)_\eta = (\nabla_V^{(\alpha)} (W - W^R))_\eta + (\nabla_V^{(\alpha)} W^R)_\eta = (\mathcal{L}_V (W - W^R))_\eta + (\nabla_V^{(\alpha)} W^R)_\eta,$$

where \mathcal{L}_V denotes the Lie derivative in the direction of any vector field \tilde{V} such that $\tilde{V}_\eta = V$. Indeed, if Z is a vector fields satisfying $Z_\eta = 0$, then locally

$$(\nabla_V Z)_\eta = DZ(\eta) \cdot V - \Gamma_\eta(Z_\eta, V) = DZ(\eta) \cdot V = [V, Z]_\eta = (\mathcal{L}_V Z)_\eta.$$

Using (3.1) and (3.3) we can compute the affine connection $\nabla^{(\alpha)}$ for arbitrary fields on $\mathcal{D}(M)$ and this completes the proof of (i).

As in the proof of Theorem 2.1, (ii) can be established by a direct calculation or, alternatively, it can be deduced from the general properties of divergences of the type (1.5) and (1.6).

Regarding (iii) let $\gamma(t)$ be a curve in $\mathcal{D}(M)$ with $\gamma(0) = e$ and $\dot{\gamma} = d\gamma/dt = u \circ \gamma$. In the appendix we prove that (3.3) implies

$$(3.4) \quad \nabla_{\dot{\gamma}}^{(\alpha)} \dot{\gamma} = u_t \circ \gamma + \nabla_{\dot{\gamma}}^{(\alpha)} \dot{\gamma}^R.$$

Using (3.1) and (3.4) the geodesic equation $\nabla_{\dot{\gamma}}^{(\alpha)} \dot{\gamma} = 0$ can be written as

$$u_t = (\Delta^{-1} df)^\sharp \quad \text{where} \quad f = d \operatorname{div} u \cdot u + \frac{1-\alpha}{2} (\operatorname{div} u)^2$$

so that setting $\varphi = \operatorname{div} u$ and taking the exterior derivative we get

$$-d\varphi_t + \delta du_t^\flat = d\iota_u d\varphi + (1-\alpha)\varphi d\varphi.$$

The relation $u_t^\flat = \Delta^{-1} df$ implies $\Delta du_t^\flat = d\Delta u_t^\flat = 0$ hence, in particular, $\delta du_t^\flat = 0$. This proves (3.2). \square

Finally, we turn to integrability of the geodesic equations in (3.2) which to the best of our knowledge has not been studied in the literature before, with the exception of the case $\alpha = 0$ in [5].

Corollary 3.3. *The geodesic equations of $\nabla^{(\alpha)}$ on $\mathcal{D}(M)/\mathcal{D}_\mu(M)$ corresponding to (3.2) with $\alpha = 0, \pm 1$ are integrable equations in any dimension n .*

Proof. Equation (3.2) with $\alpha = 0$ was derived and shown to be integrable in [5]. Since $\nabla^{(1)}$ and $\nabla^{(-1)}$ are flat, integrability of the other two equations can be established similarly to their one-dimensional analogues in Section 2. We will consider the more complicated case $\alpha = 1$ with the corresponding equation

$$(3.5) \quad d\varphi_t + d\iota_u d\varphi = 0, \quad \varphi = \operatorname{div} u.$$

Proceeding as in the proof of Theorem 2.6 we first show that the map

$$\eta \mapsto \phi(\eta) = \log \operatorname{Jac}_\mu \eta - \int_M \log \operatorname{Jac}_\mu \eta d\mu$$

from $\mathcal{D}(M)/\mathcal{D}_\mu(M)$ to the space of smooth mean-zero functions on M defines an affine chart for $\nabla^{(1)}$. Thus, as before, the geodesics of $\nabla^{(1)}$ in the affine coordinates defined by ϕ are the straight lines

$$t \rightarrow \tilde{\eta}(t, x) = a(x)t + b(x) \quad x \in M$$

where a and b are smooth mean-zero functions on M . From this we find

$$\text{Jac}_\mu \eta(t, x) = \frac{e^{a(x)t+b(x)}}{\int_M e^{a(x)t+b(x)} d\mu} \quad x \in M$$

and combining this expression with the identity $\frac{d}{dt} \text{Jac}_\mu(\eta) = (\varphi \circ \eta) \text{Jac}_\mu \eta$, we obtain

$$(3.6) \quad \varphi(t, \eta(t, x)) = a(x) - \frac{\int_M a(x) e^{a(x)t+b(x)} d\mu}{\int_M e^{a(x)t+b(x)} d\mu}.$$

Note that the time derivative of $\varphi \circ \eta$ is independent of x , so that

$$d\varphi_t + d\iota_u d\varphi = d((\varphi \circ \eta)_t \circ \eta^{-1}) = 0$$

which shows that (3.6) solves the equations in (3.5). \square

APPENDIX A. PROOF OF (3.4)

Let $\gamma(t)$ be a curve in $\mathcal{D}(M)$ with $\dot{\gamma}(t) = u(t) \circ \gamma(t)$. For any $t_0 > 0$ we compute

$$(A.1) \quad \begin{aligned} u_t(t_0) \circ \gamma(t_0) &= \left\{ \frac{d}{dt} \Big|_{t=t_0} (\dot{\gamma}(t) \circ \gamma(t)^{-1}) \right\} \circ \gamma(t_0) \\ &= \ddot{\gamma}(t_0) + D(\dot{\gamma}(t_0)) \cdot \left\{ \frac{d}{dt} \Big|_{t=t_0} \gamma(t)^{-1} \right\} \circ \gamma(t_0). \end{aligned}$$

But differentiating the relation $\gamma(t)^{-1} \circ \gamma(t) = id$ with respect to t we find

$$\left\{ \frac{d}{dt} \Big|_{t=t_0} \gamma(t)^{-1} \right\} \circ \gamma(t_0) + D(\gamma(t_0)^{-1}) \circ \gamma(t_0) \cdot \dot{\gamma}(t_0) = 0.$$

Thus, (A.1) yields

$$\begin{aligned} u_t(t_0) \circ \gamma(t_0) &= \ddot{\gamma}(t_0) - D(\dot{\gamma}(t_0)) \cdot D(\gamma(t_0)^{-1}) \circ \gamma(t_0) \cdot \dot{\gamma}(t_0) \\ &= \frac{d}{dt} \Big|_{t=t_0} \{ \dot{\gamma}(t) - \dot{\gamma}(t_0) \circ \gamma(t_0)^{-1} \circ \gamma(t) \} = D(\dot{\gamma} - \dot{\gamma}^R) \cdot \dot{\gamma}(t_0). \end{aligned}$$

Equation (3.4) now follows from (3.3).

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